Extension of Volterra Analysis to Weakly Nonlinear Electromagnetic Field Problems with Application to Whistler Propagation

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Abstract —**In this study, Volterra analysis is extended to weakly nonlinear electromagnetic field problems. Generalized Green's functions and their Fourier transforms are introduced. These are used to express the nth order system response explicitly in terms of the system input. Although the theory is developed in general, homogeneous media are assumed in the examples for simplicity.**

Application of the Volterra approach is illustrated by investigating whistler-mode propagation in a cold collisionless electron plasma. After defining the nonlinear differential equations for propagation at an angle θ to the uniform magnetic field, exponential probing inputs are used to generate the generalized Green's functions. The second-order responses, which are expressed in terms of the generalized Green's functions, are examined in detail. Computer programs are used to numerically evaluate the second-order response to a sinusoidally varying time function.

I. Introduction

THE VOLTERRA functional series [1] has been successfully applied both to weakly nonlinear circuit problems [2] and to nonlinearly loaded antenna problems [3], [4]. In this paper, the Volterra series approach is extended to weakly nonlinear field problems. Generalized Green's functions are defined and evaluated for the whistler-mode of propagation.

II. Generalized Green's Functions

It is sometimes useful to characterize a linear electromagnetic field problem in terms of its Green's function. In the case where both spatial and time variation are of interest, the Green's function is defined by the input–output relation

$$\underline{\underline{L}}\{\delta(\vec{r}-\vec{r}_1)\delta(t-t_1)\}=g(\vec{r},\vec{r}_1|t,t_1) \tag{1}$$

where $\underline{\underline{L}}$ is the linear system operator, $\delta(\cdot)$ is the impulse function, $g(\cdot)$ is the Green's function, $\vec{r}$ is the spatial vector, and t is time. The Green's function, therefore, is the system response to an excitation which is impulsive both in space and in time.

Consider the response $a(\vec{r}|t)$ to an arbitrary scalar input $\mathfrak{f}(\vec{r}|t)$, such that

$$\underline{\underline{L}}\{\mathfrak{f}(\vec{r}|t)\}=a(\vec{r}|t). \tag{2}$$

It is well known that the response can be expressed as

$$a(\vec{r}|t)=\int_{\vec{r}_1t_1} g(\vec{r},\vec{r}_1|t,t_1)\mathfrak{f}(\vec{r}_1|t_1)\,\underline{dr_1}\,dt_1 \tag{3}$$

where, in the analysis that follows,

$$\int_{\vec{r}_1t_1}\underline{dr_1}\,dt_1 \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\,dy_1\,dz_1\,dt_1. \tag{4}$$

A weakly nonlinear field problem may be conveniently analyzed by means of the Volterra approach, where $a_n(\vec{r}|t)$ is the nth order response and the total response $a(\vec{r}|t)$ is given by

$$a(\vec{r}|t)=\sum_{n=1}^{\infty} a_n(\vec{r}|t). \tag{5}$$

Volterra analysis is most useful when the field $a(\vec{r}|t)$ can be approximated by a small number of terms N. Then

$$a(\vec{r}|t)\approx\sum_{n=1}^{N} a_n(\vec{r}|t). \tag{6}$$

The truncated sum implies that higher order terms do not contribute significantly to the field.

The nth order portion of the field can be written in terms of the generalized Green's function of order n and the input as

$$a_n(\vec{r}|t)=\int_{\vec{r}_1t_1}\int_{\vec{r}_2t_2}\cdots\int_{\vec{r}_nt_n} g_n(\vec{r},\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t,t_1,t_2,\cdots,t_n)\cdot\prod_{p=1}^{n}\mathfrak{f}(\vec{r}_p|t_p)\,\underline{dr_p}\,dt_p \tag{7}$$

where

$$\prod_{p=1}^{n}\mathfrak{f}(\vec{r}_p|t_p)\,\underline{dr_p}\,dt_p=\mathfrak{f}(\vec{r}_1|t_1)\mathfrak{f}(\vec{r}_2|t_2)\cdots\cdot\mathfrak{f}(\vec{r}_n|t_n)\,\underline{dr_1}\,dt_1\,\underline{dr_2}\,dt_2\cdots\,\underline{dr_n}\,dt_n. \tag{8}$$

$a_n(\vec{r}|t)$ is of nth order in the sense that multiplication of the input by a constant C results in multiplication of the nth order output by C^n. Under the assumptions of spatial

Manuscript received August 25, 1981; revised February 22, 1982. This work was supported in part by the U.S. Air Force Systems Command, Rome Air Development Center, Electromagnetic Compatibility Branch, Griffiss Air Force Base, NY, under Contract F30602-79-C-0011.

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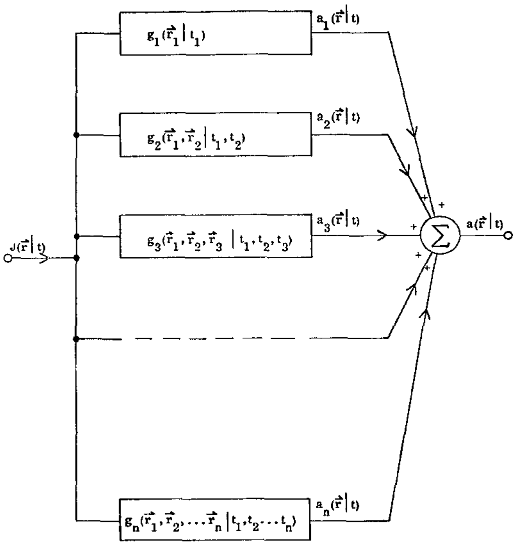

Fig. 1. Representation of a weakly nonlinear electromagnetic field problem with generalized Green's functions.

homogeneity and time invariance, the linear Green's function can be written as a function of $\vec{r}-\vec{r}_1$ and $t-t_1$. The nth order portion of the field is then given by

$$a_n(\vec{r}|t)=\int_{\vec{r}_1t_1}\int_{\vec{r}_2t_2}\cdots\int_{\vec{r}_nt_n} \cdot g_n(\vec{r}-\vec{r}_1,\vec{r}-\vec{r}_2,\cdots,\vec{r}-\vec{r}_n|t-t_1,t-t_2,\cdots,t-t_n) \cdot \prod_{p=1}^{n} \mathcal{J}(\vec{r}_p|t_p)\,\underline{dr}_p\,dt_p$$
$$=\int_{\vec{r}_1t_1}\int_{\vec{r}_2t_2}\cdots\int_{\vec{r}_nt_n} g_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t_1,t_2,\cdots,t_n) \cdot \prod_{p=1}^{n} \mathcal{J}(\vec{r}-\vec{r}_p|t-t_p)\,\underline{dr}_p\,dt_p. \quad (9)$$

A block diagram of the Volterra series model for weakly nonlinear electromagnetic field problems is shown in Fig. 1.

The Volterra kernel of (9) $g_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t_1,t_2,\cdots,t_n)$ is referred to as the generalized Green's function of order n. Its Fourier transform is defined by

$$G_n[\vec{k}_1,\vec{k}_2,\cdots,\vec{k}_n|\omega_1,\omega_2,\cdots,\omega_n] =\int_{\vec{r}_1t_1}\int_{\vec{r}_2t_2}\cdots\int_{\vec{r}_nt_n} \cdot g_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t_1,t_2,\cdots,t_n) \cdot \prod_{p=1}^{n} e^{-j(\vec{k}_p\cdot\vec{r}_p-\omega_pt_p)}\,\underline{dr}_p\,dt_p. \quad (10)$$

The transform notation that has been adopted involves the use of capital letters for functions that have been Fourier transformed from the time to the frequency domain, and the use of square brackets [·] for functions that have been transformed from the spatial to the wavenumber domain. The transform nomenclature is illustrated below:

$$g_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t_1,t_2,\cdots,t_n) \overset{\vec{r}}{\leftrightarrow} g_n[\vec{k}_1,\vec{k}_2,\cdots,\vec{k}_n|t_1,t_2,\cdots,t_n] \underset{t}{\leftrightarrow} G_n[\vec{k}_1,\vec{k}_2,\cdots,\vec{k}_n|\omega_1,\omega_2,\cdots,\omega_n].$$

$$g_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|t_1,t_2,\cdots,t_n) \underset{t}{\leftrightarrow} G_n(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_n|\omega_1,\omega_2,\cdots,\omega_n] \overset{\vec{r}}{\leftrightarrow} G_n[\vec{k}_1,\vec{k}_2,\cdots,\vec{k}_n|\omega_1,\omega_2,\cdots,\omega_n].$$

III. Class of Field Problems Where the Functional Formulation Is Appropriate

The Volterra functional series representation is particularly useful when a characterization of the system is desired such that the output due to a whole set of inputs is of interest. The technique is suitable to the analysis of weakly nonlinear electromagnetic field problems for suitable small excitations. The mild nonlinearity requirement insures that only a few generalized Green's functions are needed in order to adequately characterize the system.

In the development of the functional formulation, the concept of an analytic system was suggested [1] for systems representable by a convergent Volterra series. An analytic system is one which satisfies three properties:

1) it is deterministic—for a given input there is only one possible output;
2) it is invariant—the relationship between the input and output is independent of shifts in any of the independent variables;
3) the system cannot introduce any abrupt changes in its output, i.e., abrupt changes in the output must be due to similar abrupt changes in the input.

In general, it is not always easy to determine if a system is analytic. Ku and Wolf [5] consider the question of what class of nonlinear systems may be represented by a Volterra functional series. They suggested that nonlinear systems, represented by differential equations (where the nonlinearity can be represented by an analytic nonlinear function of the outputs and their derivatives) were suitable for the functional formulation.

The other obvious requirement for application of the functional formulation to the solution of a nonlinear electromagnetic field problem is that the linearized system be representable by a linear Green's function characterization.

IV. Multiple Input/Multiple Output Volterra Series

In (3), modeling of the input/output characteristics of a system was restricted to systems with a single input and a single output. Fields are vector quantities and the solution to an electromagnetic field problem may involve determination of more than one vector field. In addition, the excitation may be one or more vector quantities. Thus, in many problems, modeling of a field problem results in a multiple-input/multiple-output system model.

Consider the problem of modeling the system with the vector input $\overrightarrow{\mathcal{J}(\vec{r}|t)}$ and the output $\overrightarrow{a(\vec{r}|t)}$. Each component of the vector field $\vec{a}$ is assumed to be given by a Volterra functional series. The Volterra series is a generalization of the Taylor series expansion for representing a nonlinear system. The multiple input Volterra series can be written by the same sort of generalization from the multivariable Taylor series. Consider an excitation $\vec{\mathcal{J}}$, whose x-, y-, and z-components are denoted by $\mathcal{J}_x$, $\mathcal{J}_y$, and $\mathcal{J}_z$. The x-component of the vector field $\vec{a}$ can be written as

$$a_x(\vec{r}|t) = \sum_{\substack{n_1=0 \\ n\neq 0}}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \int_{\vec{r}_1 t_1} \int_{\vec{r}_2 t_2} \cdots \cdot \int_{\vec{r}_n t_n} g_{a_{(n_1,n_2,n_3)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n) \cdot \prod_{p_1=1}^{n_1} \mathcal{J}_x(\vec{r}-\vec{r}_{p_1}|t-t_{p_1})\, \underline{dr_{p_1}}\, dt_{p_1} \cdot \prod_{p_2=n_1+1}^{n_1+n_2} \mathcal{J}_y(\vec{r}-\vec{r}_{p_2}|t-t_{p_2})\, \underline{dr_{p_2}}\, dt_{p_2} \cdot \prod_{p_3=n_1+n_2+1}^{n} \mathcal{J}_z(\vec{r}-\vec{r}_{p_3}|t-t_{p_3})\, \underline{dr_{p_3}}\, dt_{p_3} \quad (11)$$

where $n = n_1 + n_2 + n_3$. By definition

$$\prod_{p=n+1}^{n} (\cdot) = 1.$$

Note that (11) contains three linear Green's functions

$$g_{a_{(1,0,0)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(0,1,0)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(0,0,1)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$

six second-order generalized Green's functions

$$g_{a_{(1,1,0)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(1,0,1)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(0,1,1)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(2,0,0)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(0,2,0)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$
$$g_{a_{(0,0,2)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n)$$

plus higher order generalized Green's functions.

The multiple-input Volterra series can also be written as the sum of terms of order n

$$a_x(\vec{r}|t) = \sum_{n=1}^{\infty} a_{nx}(\vec{r}|t) \quad (12)$$

where for three inputs, the nth order response is given by

$$a_{nx}(\vec{r}|t) = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} a_{(n-m_1-m_2, m_1, m_2)x}(\vec{r}|t) \quad (13)$$

and for $n = n_1 + n_2 + n_3$

$$a_{(n_1,n_2,n_3)x}(\vec{r}|t) = \int_{\vec{r}_1 t} \int_{\vec{r}_2 t} \cdots \int_{\vec{r}_n t_n} g_{a_{(n_1,n_2,n_3)x}}(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n | t_1, t_2, \cdots, t_n) \cdot \prod_{p_1=1}^{n_1} \mathcal{J}_x(\vec{r}-\vec{r}_{p_1}|t-t_{p_1})\, \underline{dr_{p_1}}\, dt_{p_1} \cdot \prod_{p_2=n_1+1}^{n_1+n_2} \mathcal{J}_y(\vec{r}-\vec{r}_{p_2}|t-t_{p_2})\, \underline{dr_{p_2}}\, dt_{p_2} \cdot \prod_{p_3=n_1+n_2+1}^{n} \mathcal{J}_z(\vec{r}-\vec{r}_{p_3}|t-t_{p_3})\, \underline{dr_{p_3}}\, dt_{p_3}. \quad (14)$$

In summary, the Volterra approach to the multiple-input/multiple-output nonlinear field problem is to assume that each output can be written in terms of a Volterra series. The nth order portion of the output field is written in terms of the sum of $4n$-fold integrations involving generalized Green's functions of order n and the inputs.

V. Generation of Generalized Green's Functions

In the Volterra analysis of nonlinear systems, exponential probing inputs prove quite useful. For electromagnetic fields problems, the input is chosen to be in the form of the sum of plane waves. The choice of a complex exponential excitation is simply a mathematical artifice and in no way restricts the validity of the solution since the Green's functions are independent of the form of the input. The Mth order generalized Green's function can be found by selecting the input to be the sum of M plane waves. For the single-input problem, the excitation is given by

$$\mathcal{J}(\vec{r}|t) = \sum_{m=1}^{M} e^{j(\vec{k}_m \cdot \vec{r} - \omega_m t)}. \quad (15)$$

Using the definition of the Volterra response in (5) and (9) and the definition of the Fourier transform in (10), the output field is written as

$$a(\vec{r}|t) = \sum_{n=1}^{\infty} \sum_{m_1=1}^{M} \sum_{m_2=1}^{M} \cdots \cdot \sum_{m_n=1}^{M} G_n\left[\vec{k}_{m_1}, \vec{k}_{m_2}, \cdots, \vec{k}_{m_n} | \omega_{m_1}, \omega_{m_2}, \cdots, \omega_{m_n}\right] \cdot e^{j[(\vec{k}_{m_1} + \vec{k}_{m_2} + \cdots \vec{k}_{m_n}) \cdot \vec{r} - (\omega_{m_1} + \omega_{m_2} + \cdots \omega_{m_n})t]}. \quad (16)$$

The input and output can then be substituted into the nonlinear differential equations describing the problem. Using the linear independence property of spatial and/or time exponentials it is possible to separate out the algebraic equation containing only terms of Mth order with exponential behavior given by

$$e^{j[(\vec{k}_1+\vec{k}_2+\cdots\vec{k}_M)\cdot\vec{r}-(\omega_1+\omega_2+\cdots\omega_M)t]}.$$

The resultant equation will contain generalized Green's function transforms of order M plus terms of products of lower order transforms of generalized Green's functions. The sum of the orders of all terms in a product will be equal to M. Thus, the Mth-order generalized Green's function transforms can be written in terms of lower order generalized Green's functions transforms and the solution procedure can be considered to be a "bootstrap" operation. Starting with the linear Green's functions, as many generalized Green's functions as are needed to adequately characterize the nonlinearity can be generated.

Once the generalized Green's functions in the waveform/frequency domain have been obtained, inverse Fourier transform techniques are used to write generalized Green's functions in the spatial and/or time domain as required.

VI. Simple One-Dimensional Example

An example used in many text books on elementary electromagnetic field theory to demonstrate the use of Gauss's law involves an infinite plate of constant surface charge density σ_S located in the $x-y$ plane at $z=z_0$. Using Gauss's law in integral form and the usual symmetry arguments, the electric flux density $\vec{\mathfrak{D}}$ is given by [13]

$$\vec{\mathfrak{D}}=\sigma_S\left[U(z-z_0)-\tfrac{1}{2}\right]\hat{z} \tag{17}$$

where $U(\cdot)$ is the unit step function.

Now, assume a medium with a nonlinear relationship between the scalar electric field intensity and electric flux density given by

$$\mathfrak{D}_z=\epsilon\mathcal{E}_z+\alpha\mathcal{E}_z^2 \tag{18}$$

where ϵ and α are constants. For $z>z_0$, and for small α (mild nonlinearity assumption), the electric field intensity can be shown to be given by

$$\begin{aligned}\mathcal{E}_z&=\frac{\epsilon}{2\alpha}\left[-1+\left(1+\frac{2\alpha\sigma_S}{\epsilon^2}\right)^{1/2}\right]\\&=\frac{\sigma_S}{2\epsilon}-\frac{\alpha}{\epsilon}\left(\frac{\sigma_S}{2\epsilon}\right)^2+2\left(\frac{\alpha}{\epsilon}\right)^2\left(\frac{\sigma_S}{2\epsilon}\right)^3-\cdots.\end{aligned} \tag{19}$$

This problem can also be solved using a Volterra functional series formulation. Gauss's law in point form is given by

$$\nabla\cdot\vec{\mathfrak{D}}=\rho \tag{20}$$

where ρ is the volume charge density. A symmetrical problem will be assumed where ρ is constrained to be a function of z only. Using this symmetry, the differential equation becomes

$$\frac{\partial}{\partial z}\mathfrak{D}_z(z)=\rho(z). \tag{21}$$

Substituting (18) into (21) yields

$$\epsilon\frac{\partial}{\partial z}\mathcal{E}_z(z)=\rho(z)-\alpha\frac{\partial}{\partial z}\mathcal{E}_z^2(z). \tag{22}$$

Assume a Volterra series solution of the form

$$\mathcal{E}_z(z)=\sum_{n=1}^{\infty}\mathcal{E}_{nz}(z) \tag{23}$$

where the nth-order z-component is given by

$$\mathcal{E}_{nz}(z)=\int_{z_1}\int_{z_2}\cdots\int_{z_n}g_n(z_1,z_2,\cdots,z_n)\prod_{p=1}^{n}\rho(z-z_p)\,dz_p. \tag{24}$$

By employing exponential probing inputs, expressions for the first three generalized Green's functions in the wavenumber domain are found to be

$$g_1[k_{1z}]=\frac{1}{jk_{1z}\epsilon} \tag{25}$$

$$\begin{aligned}g_2[k_{1z},k_{2z}]&=-\frac{\alpha}{\epsilon}g_1[k_{1z}]g_1[k_{2z}]\\&=-\left(\frac{\alpha}{\epsilon^3}\right)\left(\frac{1}{jk_{1z}}\right)\left(\frac{1}{jk_{2z}}\right)\end{aligned} \tag{26}$$

and

$$\begin{aligned}g_3[k_{1z},k_{2z},k_{3z}]&=-\frac{2\alpha}{3\epsilon}\{g_1[k_{1z}]g_2[k_{2z},k_{3z}]\\&\quad+g_1[k_{2z}]g_2[k_{1z},k_{3z}]\\&\quad+g_1[k_{3z}]g_2[k_{1z},k_{2z}]\}.\\&=\frac{2\alpha^2}{\epsilon^5}\left(\frac{1}{jk_{1z}}\right)\left(\frac{1}{jk_{2z}}\right)\left(\frac{1}{jk_{3z}}\right).\end{aligned} \tag{27}$$

Generalized Green's functions in z-space are obtained via inverse Fourier transforms. Therefore

$$g_1(z_1)=\frac{1}{\epsilon}\left[U(z_1)-\tfrac{1}{2}\right]. \tag{28}$$

$$g_2(z_1,z_2)=-\frac{\alpha}{\epsilon^3}\left[U(z_1)-\tfrac{1}{2}\right]\left[U(z_2)-\tfrac{1}{2}\right] \tag{29}$$

and

$$g_3(z_1,z_2,z_3)=\frac{2\alpha^2}{\epsilon^5}\left[U(z_1)-\tfrac{1}{2}\right]\left[U(z_2)-\tfrac{1}{2}\right]\left[U(z_3)-\tfrac{1}{2}\right]. \tag{30}$$

In this model, the infinite plate of constant surface charge density σ_S located at $z=z_0$ is representable as the input

$$\rho(z)=\sigma_S\delta(z-z_0). \tag{31}$$

The output is written in terms of generalized Green's functions by substituting (31) into (24) and then into (23) to get

$$\mathcal{E}_z(z)=\sum_{n=1}^{\infty}\sigma_S^n g_n(z-z_0,z-z_0,\cdots,z-z_0). \tag{32}$$

For $z > z_0$, the output is given by

$$\mathcal{E}_z = \frac{\sigma_S}{2\epsilon} - \frac{\alpha}{\epsilon}\left(\frac{\sigma_S}{2\epsilon}\right)^2 + 2\left(\frac{\alpha}{\epsilon}\right)^2\left(\frac{\sigma_S}{2\epsilon}\right)^3 - \cdots. \quad (33)$$

The Volterra functional series and the generalized Green's functions have been used to model a hypothetical mildly nonlinear field problem. The output field due to a specific excitation was found to agree with the more classical solution given in (19).

VII. Introduction to the Whistler Mode

Electromagnetic wave propagation through an infinite unmagnetized plasma is cut off at frequencies below the electron plasma frequency ω_p, but in a magnetoplasma, such as the ionosphere, a window is open for propagation in the frequency range below the electron cyclotron frequency ω_c. The vector wavenumber $\vec{k}$ is real in a cone-shaped region with axis along the magnetic field $\vec{B}_M$, and cone angle θ_{RES}. When the propagation direction is along the static magnetic field, the wave is right-hand circularly polarized [6], [9]. Since the plasma in which they travel is dispersive, the different frequency components of these waves are spread out in time. The whistle-like sound of detected waves of this type propagating through the ionosphere is the origin of the name "whistler". Natural whistlers are believed to originate with atmospheric disturbances, such as lightning.

The nonlinear mechanisms that affect the propagation of whistlers are in two categories. The first depends on the heating of electrons by the RF fields and the variation of the electron collision frequency with temperature. The result is a nonlinear component of current that is proportional to the third power of the field, and one effect is the generation of third harmonics. The second mechanism, which occurs at any electron temperature, is in part the radiation pressure of the wave [10]. The resulting nonlinear current is proportional to the second power of the field, and one effect is the generation of second harmonics. Although the thermal nonlinearity is the larger at modest field intensities [11], there are some interesting aspects of the radiation pressure effect that merit study of this nonlinearity.

Harmonic generation in plasmas is generally a weak effect because the dispersion is not compatible with the harmonic relationship of wavenumbers, i.e., the resonance condition [12]. In the case of the whistler mode, however, the resonance condition for the second harmonic and the dispersion relation can be satisfied simultaneously for certain frequencies and cone angle. Under these conditions, the second harmonic is greatly enhanced.

The whistler mode propagating parallel to the magnetic field is an exact solution to the field problem including the electromagnetic nonlinearities. An interesting question to be answered is how the nonlinear interaction products vary with cone angle.

VIII. Nonlinear Differential Equation Formulation for the Whistler

The whistler is assumed to be propagating in an infinite, cold, collisionless, electron plasma that is uniformly magnetized. On the average, charge neutrality is assumed with a background of immobile positively charged ions of density n_0 ions per cubic meter, and a collection of mobile negatively charged electrons having an average density also equal to n_0. The set of vector nonlinear differential equations governing whistler propagation are

$$\nabla \times \vec{\mathcal{E}} + \mu_0 \frac{\partial \vec{\mathcal{H}}}{\partial t} = 0 \quad (34)$$

$$\nabla \times \vec{\mathcal{H}} - \epsilon_0 \frac{\partial \vec{\mathcal{E}}}{\partial t} = -en_0\vec{v} + \epsilon_0(\nabla \cdot \vec{\mathcal{E}})\vec{v} + \vec{\mathcal{J}}_I \quad (35)$$

$$\frac{\partial \vec{v}}{\partial t} + \eta\vec{\mathcal{E}} + \vec{v} \times \vec{\omega}_c = -(\vec{v} \cdot \nabla)\vec{v} - \eta\mu_0\vec{v} \times \vec{\mathcal{H}} \quad (36)$$

where $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$, and $\vec{v}$ are the time and spatially varying electric field intensity, magnetic field intensity, and velocity vectors; $\vec{\omega}_c$ is the electron vector cyclotron frequency; $\vec{\mathcal{J}}_I$ is the impressed current density; and η is the electron charge to mass ratio. Note the nonlinear terms, $\epsilon_0(\nabla \cdot \vec{\mathcal{E}})\vec{v}$, $-(\vec{v} \cdot \nabla)\vec{v}$, and $-\eta\mu_0\vec{v} \times \vec{\mathcal{H}}$ in (34)–(36).

The problem to be investigated involves the nonlinear whistler propagation in the direction of $\vec{k}$ at an angle θ with respect to the static magnetic flux density $\vec{B}_M$. To explore solutions which are uniform waves propagating in the z-direction (i.e., $\vec{k} = k_z\hat{z}$), it is convenient to write

$$\vec{\mathcal{E}} = \hat{x}\mathcal{E}_x(z|t) + \hat{y}\mathcal{E}_y(z|t) + \hat{z}\mathcal{E}_z(z|t) \quad (37)$$

$$\vec{\mathcal{H}} = \hat{x}\mathcal{H}_x(z|t) + \hat{y}\mathcal{H}_y(z|t) + \hat{z}\mathcal{H}_z(z|t) \quad (38)$$

and

$$\vec{v} = \hat{x}v_x(z|t) + \hat{y}v_y(z|t) + \hat{z}v_z(z|t). \quad (39)$$

In order to excite the circularly polarized waves of the whistler mode, Lee [6] applies current sheets using dense arrays of filamentary conductors perpendicular to each other and to the direction of propagation, such that

$$\vec{\mathcal{J}}_I = \hat{x}\,\delta(z)\mathcal{J}_x(t) + \hat{y}\,\delta(z)\mathcal{J}_y(t). \quad (40)$$

The impressed current source in this analysis is assumed to be of the more general form given by

$$\vec{\mathcal{J}}_I = \hat{x}\mathcal{J}_{I_x}(z|t) + \hat{y}\mathcal{J}_{I_y}(z|t). \quad (41)$$

IX. Generation of Generalized Green's Functions in the Transform Domain

The nonlinear problem as described in (37), (38), and (39) involves a system with the two inputs—$\mathcal{J}_x$ and $\mathcal{J}_y$—and nine outputs—the three components of the three vector fields. A Volterra series solution is given by

$$a_\xi(z|t) = \sum_{n=1}^{\infty} a_{n\xi}(z|t) \quad (42)$$

for $a = \mathcal{E}$, $\mathcal{H}$, and v and for $\xi = x$, y, and z. The nth order portion of the fields due to the two excitations is given by

$$a_{n\xi}(z|t) = \sum_{n_1=0}^{n} a_{(n_1, n-n_1)\xi}(z|t) \tag{43}$$

where

$$a_{(n_1, n-n_1)\xi}(z|t) = \int_{z_1 t_1} \int_{z_2 t_2} \cdots \int_{z_n t_n} g_{a(n_1, n-n_1)\xi} \cdot (z_1, z_2, \cdots, z_n | t_1, t_2, \cdots, t_n) \cdot \prod_{p_1=1}^{n} \mathcal{J}_{I_x}(z - z_{p_1} | t - t_{p_1})\, dz_{p_1} dt_{p_1} \cdot \prod_{p_2=n_1+1}^{n} \mathcal{J}_{I_y}(z - z_{p_2} | t - t_{p_2})\, dz_{p_2} dt_{p_2}. \tag{44}$$

Note, for each of the nine outputs there are two first-order terms $a_{(1,0)\xi}(z|t)$ and $a_{(0,1)\xi}(z|t)$ corresponding to inputs $\mathcal{J}_{I_x}(z|t)$ and $\mathcal{J}_{I_y}(z|t)$, respectively.

Using plane-wave probing inputs, the nine dimensional vectors of linear Green's functions in the wave-number/frequency domain are given by

$$\overrightarrow{G_{(1-m,m)}[k_z|\omega]} = \underline{D[k_z|\omega]}^{-1}\, \overrightarrow{J_{(1-m,m)}} \tag{45}$$

for $m = 0$ and 1; where

$$\overrightarrow{J_{(1-m,m)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1-m \\ m \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

and

$$\underline{D[k_z|\omega]} = \left[\begin{array}{ccc|ccc|ccc} 0 & -jk_z & 0 & -j\omega\mu_0 & 0 & 0 & 0 & 0 & 0 \\ jk_z & 0 & 0 & 0 & -j\omega\mu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j\omega\mu_0 & 0 & 0 & 0 \\ \hline j\omega\epsilon_0 & 0 & 0 & 0 & -jk_z & 0 & en_0 & 0 & 0 \\ 0 & j\omega\epsilon_0 & 0 & jk_z & 0 & 0 & 0 & en_0 & 0 \\ 0 & 0 & j\omega\epsilon_0 & 0 & 0 & 0 & 0 & 0 & en_0 \\ \hline \eta & 0 & 0 & 0 & 0 & 0 & -j\omega & \omega_{c_z} & -\omega_{c_y} \\ 0 & \eta & 0 & 0 & 0 & 0 & -\omega_{c_z} & -j\omega & \omega_{c_x} \\ 0 & 0 & \eta & 0 & 0 & 0 & \omega_{c_y} & -\omega_{c_x} & -j\omega \end{array}\right]. \tag{47}$$

In the Volterra series, formulation of the nonlinear whistler problem, the Mth-order portion of a field component, as given in (43), contains $M+1$ terms. Thus, characterization of the Mth-order behavior involves determination of $(M+1)$, Mth order generalized Green's functions for each of the field quantities of interest. The Mth-order generalized Green's functions in the transform domain are given by

$$\overrightarrow{G_{(M-m,m)}[k_{1z}, k_{2z}, \cdots, k_{mz} | \omega_1, \omega_2, \cdots, \omega_M]} = \underline{D[k_z^{\Sigma} | \omega^{\Sigma}]}^{-1}\, \overrightarrow{J_{(M-m,m)}[k_{1z}, k_{2z}, \cdots, k_{mz} | \omega_1, \omega_2, \cdots, \omega_M]}. \tag{48}$$

For $m = 0, 1, \cdots$, and M, where

$$k_z^{\Sigma} = \sum_{n=1}^{M} k_{nz} \tag{49}$$

and

$$\omega^{\Sigma} = \sum_{n=1}^{M} \omega_n. \tag{50}$$

The components of the Mth-order driving term $J_{(M-m,m)}[\cdot]$ are made up of the sums of products of lower order transforms of the generalized Green's functions.

X. Solution of Nonlinear Whistler Problem for Input That is Sinusoidal in Time and Impulsive in Space

In order to excite the right-hand circularly polarized waves of the whistler mode in the laboratory, Lee [6] suggests the use of dense arrays of filamentary conductors perpendicular to each other. For a wave propagating in the $+z$-direction, the impressed current density is of the form

$$\overrightarrow{\mathcal{J}_I}(z|t) = c_1\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\}\,\delta(z). \tag{51}$$

The vector of nine linear output components $\overrightarrow{a_1(z|t)}$ is given by

$$\overrightarrow{a_1(z|t)} = \tfrac{1}{2}c_1\left\{\overrightarrow{G_{(1,0)}(z|-\omega_0)} - j\,\overrightarrow{G_{(0,1)}(z|-\omega_0)}\right\}e^{j\omega_0 t} + \tfrac{1}{2}c_1\left\{\overrightarrow{G_{(1,0)}(z|\omega_0)} + j\,\overrightarrow{G_{(0,1)}(z|\omega_0)}\right\}e^{-j\omega_0 t}. \tag{52}$$

A linear Green's function in the spatial/frequency domain $G_1(z|\omega)$ is related to $G_1[k_z|\omega]$ via the inverse Fourier transform

$$G_1(z|\omega)=\frac{1}{2\pi}\int_{k_z} G_1[k_z|\omega]e^{jk_z z}\,dk_z. \quad (53)$$

For the input given in (51), the vector of nine second-order output field components $\overrightarrow{a_2(z|t)}$ is given by

$$\begin{aligned}\overrightarrow{a_2(z|t)} &= \tfrac{1}{4}c_1^2\{\overrightarrow{G_{(2,0)}(z,z|-\omega_0,-\omega_0)} \\ &- j\overrightarrow{G_{(1,1)}(z,z|-\omega_0,-\omega_0)} \\ &- \overrightarrow{G_{(0,2)}(z,z|-\omega_0,-\omega_0)}\}e^{j2\omega_0 t} \\ &+\tfrac{1}{4}c_1^2\{\overrightarrow{G_{(2,0)}(z,z|\omega_0,\omega_0)}+j\overrightarrow{G_{(1,1)}(z,z|\omega_0,\omega_0)} \\ &- \overrightarrow{G_{(0,2)}(z,z|\omega_0,\omega_0)}\}e^{-j2\omega_0 t} \\ &+\tfrac{1}{4}c_1^2\{2\overrightarrow{G_{(2,0)}(z,z|-\omega_0,\omega_0)}+j\overrightarrow{G_{(1,1)}(z,z|-\omega_0,\omega_0)} \\ &- j\overrightarrow{G_{(1,1)}(z,z|\omega_0,-\omega_0)}-2\overrightarrow{G_{(0,2)}(z,z|-\omega_0,\omega_0)}\}. \quad (54)\end{aligned}$$

Evaluation of the output requires knowledge of the generalized Green's functions in the spatial/frequency domain for specific values of frequency. A second-order generalized Green's function in the spatial/frequency domain $G_2(z_1,z_2|\omega_1,\omega_2)$ is related to $G_2[k_{1z},k_{2z}|\omega_1,\omega_2]$ via the two-fold inverse Fourier transform relationship

$$G_2(z_1,z_2|\omega_1,\omega_2)=\frac{1}{(2\pi)^2}\int_{k_{1z}}\int_{k_{2z}} G_2[k_{z_1},k_{z_2}|\omega_1,\omega_2] \cdot\prod_{p=1}^{2} e^{jk_{pz}z_p}\,dk_{pz}. \quad (55)$$

In particular

$$G_2(z,z|\omega_1,\omega_2)=G_2(z_1,z_2|\omega_1,\omega_2)|_{z_1=z_2=z} \quad (56)$$

is desired for specific values of ω_1 and ω_2 ($\pm\omega_0$).

The above procedure requires performing a two-dimensional inverse transform followed by an association of spatial variables (i.e., $z_1=z_2=z$). A more attractive procedure is to associate variables in the transform or wavenumber domain such that only a single dimensional inverse Fourier transform is required. This association of variables in the transform domain was originally developed by George [7]. George's technique is applied to the evaluation of $G_2(z,z|\omega_1\omega_2)$ for the nonlinear whistler problem in Dalpe [8].

XI. Numerical Comparison of the First- and Second-Order Plane-Wave Responses

A plane wave propagating in the $+z$ direction with angular frequency ω has the form $e^{j(k_z z-\omega t)}$. The linear plane-wave response with angular frequency ω_0 is now compared to the second-order plane-wave response at angular frequency $2\omega_0$.

From (52), that portion of the linear response that can contribute plane waves propagating in the $+z$-direction with angular frequency ω_0 is given by

$$\tfrac{1}{2}c_1\{\overrightarrow{G_{(1,0)}(z|\omega_0)}+j\overrightarrow{G_{(0,1)}(z|\omega_0)}\}e^{-j\omega_0 t}. \quad (57)$$

Likewise, from (54), the part of the second-order response that can contribute plane waves propagating in the $+z$-direction with angular frequency $2\omega_0$ is given by

$$\tfrac{1}{4}c_1^2\{\overrightarrow{G_{(2,0)}(z,z|\omega_0,\omega_0)}+j\overrightarrow{G_{(1,1)}(z,z|\omega_0,\omega_0)} - \overrightarrow{G_{(0,2)}(z,z|\omega_0,\omega_0)}\}e^{-j2\omega_0 t}. \quad (58)$$

The generalized Green's functions in the wavenumber domain for constant ω can be written in terms of expansions in terms of the poles in k_z-space. The required z-space expressions are given by the corresponding inverse Fourier transforms. Plane-wave solutions of the form desired correspond to positive real poles in k_z-space. The positive real poles yield plane waves of the form desired. Solution of the linear dispersion relation (Appleton–Hartree dispersion relation [9])

$$n^2\equiv\frac{k_z^2}{\omega^2\mu_0\epsilon_0}=1-\frac{\omega_p^2}{\omega^2-\dfrac{\omega_c^2\omega^2\sin^2\theta}{2(\omega^2-\omega_p^2)}\pm\left[\dfrac{\omega_c^4\omega^4\sin^4\theta}{4(\omega^2-\omega_p^2)^2}+\omega^2\omega_c^2\cos^2\theta\right]^{1/2}} \quad (59)$$

for propagation in an infinite homogeneous collisionless plasma with $\omega=\omega_0$ yields the poles in k_z-space. For whistler mode propagation, only one real positive pole exists (p_1). The second-order generalized Green's functions in the wavenumber/frequency domain contain two poles in k_z-space for $\omega_1=\omega_2=\omega$ that can be real and greater than zero. The pole at $2p_1$ is positive-real for all ω_0 where the linear whistler propagates. The pole at q_1 corresponds to the positive-real root of k_z in the linear dispersion relation for $\omega=2\omega_0$ (if it exists). The second-order response due to the sinusoidally varying time function given in (53) contains terms with exponential behavior given by $e^{j(2p_1z-2\omega_0 t)}$ and $e^{j(q_1z-2\omega_0 t)}$. Note that the second-order response contains two plane waves with different phase velocities due to the different propagation constants ($2p_1$ and q_1). A computer program was written to numerically evaluate the linear and second-order plane-wave responses. In Figs. 2 through 5 the magnitude of the linear response and the magnitudes of the ratio of the second-order responses to the linear response are plotted for the x-component of the $\mathcal{E}$ field.

As was stated earlier, plane-wave propagation corresponds to real poles in k_z-space. Real poles exist for cone angles less than the resonance cone angle θ_{RES}. As θ approaches θ_{RES}, the real poles of the whistler in k_z-space approach infinity. The equation for the resonance cone is given by

$$\tan^2\theta_{RES}=\frac{(\omega_p^2-\omega^2)(\omega_c^2-\omega^2)}{\omega^2(\omega_p^2+\omega_c^2-\omega^2)}. \quad (60)$$

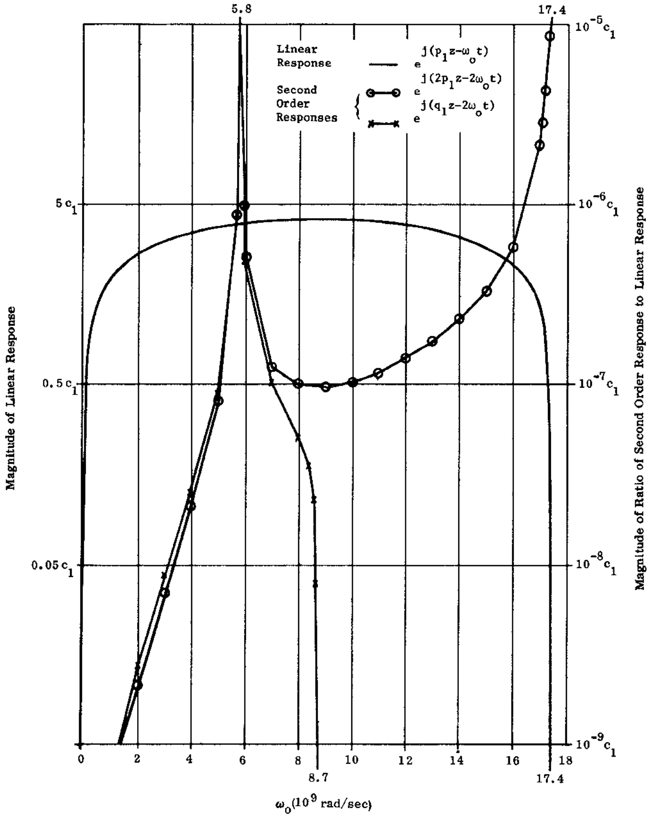

Fig. 2. Plane-wave responses for $\omega_p = 100\times10^9$ rad/s, $\omega_c = 25\times10^9$ rad/s, and $\theta = 45°$ as a function of the excitation angular frequency ω_0 for $\mathcal{E}_x(z|t)$.

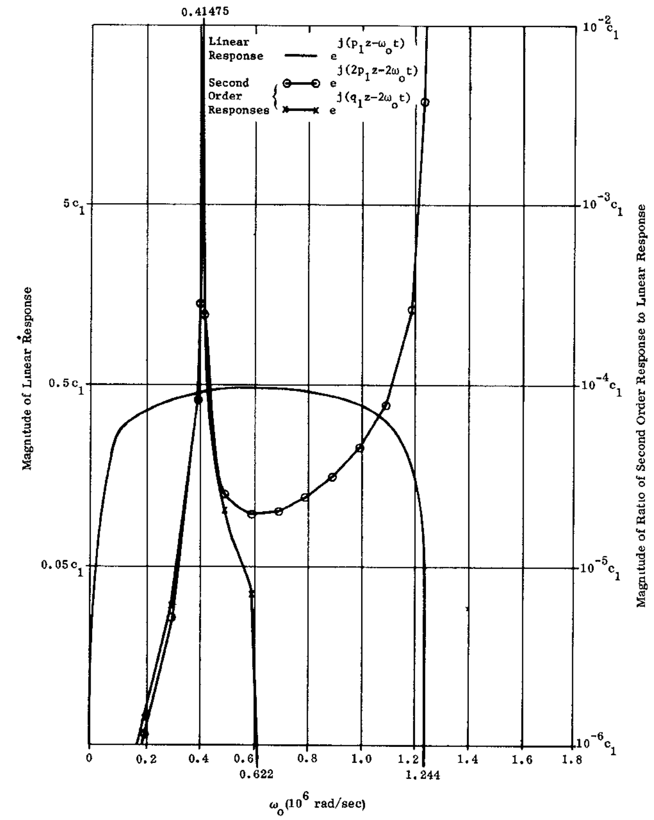

Fig. 4. Plane-wave responses for $\omega_p = 60\times10^6$ rad/s, $\omega_c = 1.76\times10^6$ rad/s, and $\theta = 45°$ as a function of the excitation angular frequency ω_0 for $\mathcal{E}_x(z|t)$.

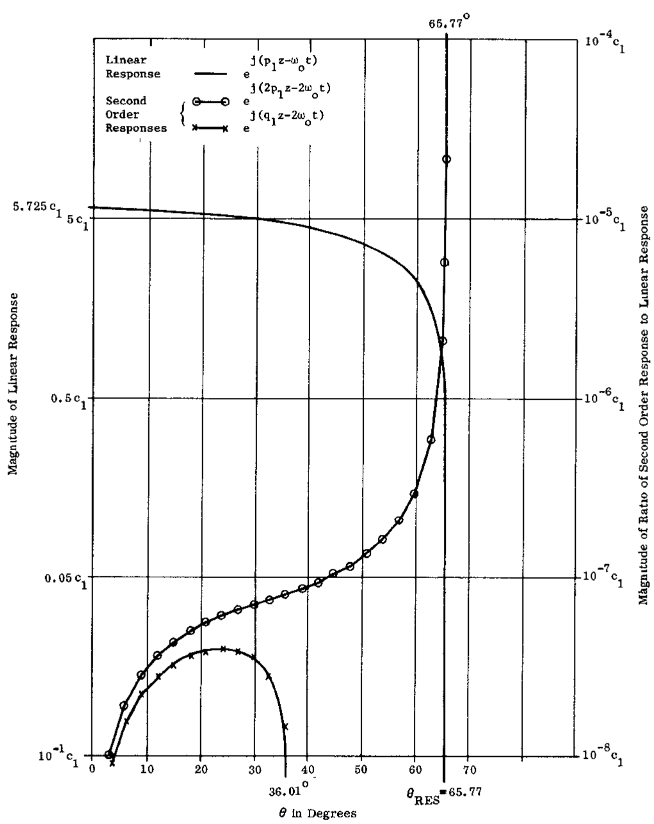

Fig. 3. Plane-wave responses for $\omega_p = 100\times10^9$ rad/s, $\omega_c = 25\times10^9$ rad/s, and $\omega_0 = 10\times10^9$ rad/s as a function of the angle between the vector wavenumber and the static magnetic field for $\mathcal{E}_x(z|t)$.

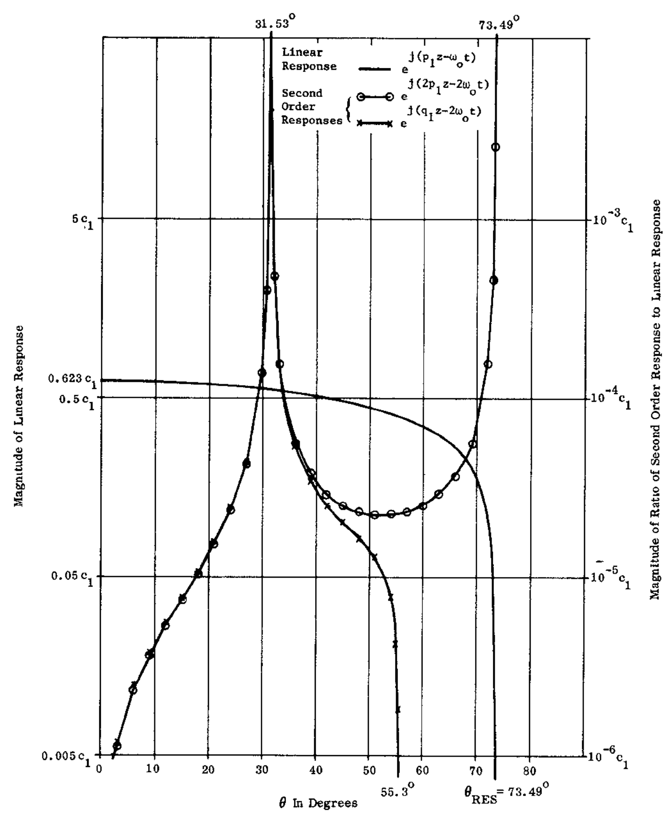

Fig. 5. Plane-wave responses for $\omega_p = 60\times10^6$ rad/s, $\omega_c = 1.76\times10^6$ rad/s, and $\omega_0 = 0.5\times10^6$ rad/s as a function of the angle between the vector wavenumber and the static magnetic field for $\mathcal{E}_x(z|t)$.

In Figs. 2 and 4, the magnitudes of the plane-wave responses are plotted as a function of the excitation frequency ω_0 for $\theta = 45°$. As ω_0 increases, the resonance condition (60) is approached and the poles in k_z-space approach infinity and become imaginary for larger values of ω_0. The wavenumber p_1 of the linear plane wave is determined by the pole location for $\omega = \omega_0$. Thus, the linear plane wave and, consequently, the second-order plane wave with wavenumber $2p_1$ are cut off for

$$\omega_0 > \left[\frac{\omega_c^2 + \omega_p^2 - \left(\omega_c^4 + \omega_p^4\right)^{1/2}}{2}\right]^{1/2}. \qquad (61)$$

The wavenumber q_1 of the second-order plane wave is determined by the pole location in k_z-space for $\omega = 2\omega_0$. This plane wave will cut off for

$$\omega_0 > \tfrac{1}{2}\left[\frac{\omega_c^2 + \omega_p^2 - \left(\omega_c^4 + \omega_p^4\right)^{1/2}}{2}\right]^{1/2}. \qquad (62)$$

The expansion of the second-order Green's functions in terms of their k_z-space poles has assumed that the poles at $2p_1$ and q_1 are distinct. Should $2p_1 = q_1$, a second-order pole exists and the result is a term whose amplitude is linear in z. Under these conditions, the convergence of the Volterra series is questionable except for small z, but the unbounded nature of the term can be removed by adding a small amount of damping. The condition $2p_1 = q_1$ means that both first- and second-order responses are natural modes for propagation, and the effect of the nonlinear coupling is enhanced. Given that p_1 is a real pole for $\omega = \omega_0$, the condition under which $2p_1$ is also a real pole for $\omega = 2\omega_0$ can be derived from the linear dispersion relation. For $\omega_p > \omega_0$, it is met when

$$3\omega_0^2\left\{1 - \frac{\omega_c^2 \sin^2\theta}{2\left(\omega_0^2 - \omega_p^2\right)}\right\} = \omega_0\omega_c\cos\theta\left\{2\left[1 + \frac{\omega_c^2\omega_0^2\tan^2\theta\sin^2\theta}{\left(\omega_0^2 - \omega_p^2\right)^2}\right]^{1/2} - \left[1 + \frac{\omega_c^2\omega_0^2\tan^2\theta\sin^2\theta}{4\left(\omega_0^2 - \omega_p^2\right)^2}\right]^{1/2}\right\}. \qquad (63)$$

If it is assumed that $\omega_p \gg \omega_0$, then for a large range of θ less than 90°, (63) can be approximated by

$$\omega_0 \approx \frac{\omega_c \cos\theta}{3}. \qquad (64)$$

In the ionosphere, the thermal nonlinearity, which generates a third harmonic, is generally dominant [11] and little consideration has been given to the cold plasma mechanisms treated here. Nevertheless, the relation given by (64) may be satisfied for the second harmonic when a similar relation for the third harmonic cannot be satisfied. Then, the second-harmonic generation should dominate.

XII. Conclusions

The Volterra functional series is a powerful tool in the analysis of mildly nonlinear systems. A great deal of work has been done by other investigators in developing the theory as it applies to circuit type problems (problems where the linear behavior is characterized by linear impulse response $h(t)$). The major emphasis of this work has been the logical extension of the theory to systems whose linear behavior is characterized by Green's functions. A set of generalized Green's functions was used to characterize mildly nonlinear electromagnetic field problems and to predict system responses. Application of the Volterra approach was illustrated by investigating whistler-mode propagation in a cold collisionless electron plasma.

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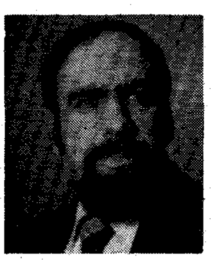

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A Study of the Noise-Temperature Performance of a Satellite Communications Low-Noise Amplifier Subsystem

MITSUGI KAJIKAWA

***Abstract*—The heart of the low-noise amplifier (LNA) subsystem is the parametric amplifier which consists of a parametric amplifier proper and a circulator. The LNA subsystem can be simplified into an equivalent circuit, to evaluate its noise-temperature performance by assuming that it consists of a parametric amplifier proper and a circulator, which in its overall sense includes the additional components of the input line as elements in one arm of the circulator. Using this simplified equivalent circuit, the noise-temperature performance is analyzed theoretically and provides a precise value for the LNA subsystem noise-temperature degradation, the noise-temperature increase of the earth-station receiving system caused by connecting an actual antenna to the subsystem and the measurement error of the HOT/COLD load noise temperature measurement method.**

Manuscript received October 22, 1981; revised February 9, 1982.

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I. INTRODUCTION

IN A SATELLITE communications earth station, the low-noise amplifier (LNA) subsystem is one of the most important subsystems and its noise temperature performance makes a major contribution to the figure of merit

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